where $C$ is a constant depending on $N$ only[3] and $b$ is the breadth of the plate. Equating Eqs. (7) and (9) yields

$$2(N+1)C=\alpha\int_{-\infty}^{\infty}\left[1-\frac{N-1}{N+1}\left(\frac{1}{2C^{N-1}}\right)^{1/N}|\eta|N+1/N\right]^{N/N-1}d\eta \qquad (10)$$

For any value of $N$, $C$ can be obtained from Ref. 3, and hence $\alpha$ can be found from Eq. (10). By substituting the value of $\alpha$ into Eq. (6), the velocity function $g$ can then be obtained. For the case of $N=0.5$, $C=0.58$, Eq. (10) may be simplified to

$$3\times 0.58=d\eta/1-\alpha/12|\eta|^{3} \qquad (11)$$

Assuming that $\eta_1=(\alpha/12)^{1/3}\eta$, Eq. (11) becomes

$$3\times 0.58=(12)^{1/3}\alpha^{2/3}\int_{-\alpha}^{\alpha}d\eta_1/1-|\eta_1|^{3}=(12)^{1/3}\alpha^{2/3}\pi/\sqrt{3}$$

and therefore $\alpha$ is equal to 0.271. From Eq. (6), the velocity function $g$ becomes

$$g=1/1+0.0226|\eta|^{3} \qquad (12)$$

$g$ is plotted vs $\eta$ as shown in Fig. 1. The special case of $N=1$ is also shown.

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# Added Mass of a Rectangular Cylinder in a Rectangular Canal

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## Introduction

IN this Note the added mass of a rectangular cylinder in a rectangular canal is considered as the frequency tends to zero, i.e., with a rigid free-surface boundary condition. For small clearances between the body and the canal, an approximate formula for the added mass is obtained from simple elementary solutions in subdivided finite elements. The sway added mass of the rectangular cylinder with half-beam $a$ and draft $b$ located at the center of a rectangular canal with half-width $c$ and depth $h$, for the zero-frequency limit, is given by

$$\mu=\frac{m}{3a}\left\{\frac{bh}{c-a}+\frac{b(2a+c)}{h-b}+(c-a)\right\}$$

where $\mu$ is the sway added mass for the zero-frequency limit and $m$ is the mass of the fluid displaced by the body.

To test the validity of the previous formula, several comparisons are made with results obtained by the straightforward finite-element method based on the dual extremum principles of Bai.[1,2] This formula is shown to provide good predictions of added mass for the range

$$0<(h-b)/a\leq 0.3, \qquad 0<(c-a)/b\leq 0.3$$

The formula is also compared with a formula obtained by Newman.[3]

## Mathematical Formulation and Solution

The computation of the added mass of a cylinder in a canal or restricted water has practical importance in naval hydrodynamics, e.g., in ship maneuvering. There are several methods to compute the added mass: the method of conformal mapping, the method of Green's function, the finite-element method, or the hypercircle method, etc.

Recently the added mass in water of finite depth has been computed by the finite-element method based on the dual extremum principles of Bai.[4] Fujino[5-8] has investigated the added mass of two-dimensional cylinders in a canal for the limiting frequencies using the hypercircle method first advocated by Synge. Bai[1,2] also computed the added mass of triangular, rectangular, circular, and Lewis-form sections in a canal by the finite-element method based on the dual extremum principles. Newman[9] obtained analytic expressions for the added mass in a canal. Recently, Newman[3] also obtained an approximate formula for the added mass of a rectangular cylinder by assuming that the clearance between the body and the canal is small.

In this Note we present an approximate formula of the added mass in a canal under assumptions similar to those made by Newman.[3] By making use of a simple solution of Laplace's equation, an approxmate formula for the added mass is obtained in terms of the geometry of the body and canal for small clearances between the body and the canal walls. In spite of the extremely simple way of obtaining the approximate formula, this formula gives good approximate results for a range of small body-canal clearances.

The sway added mass $\mu$ in the zero-frequency limit can be interpreted as the zero Froude-number limit for horizontal translatory motion. This can also be interpreted as the heave added mass in the infinite frequency limit by rotating all of the boundary configurations by 90° when the geometry is symmetric with respect to the $y$ axis. However, the zero limiting-frequency problem is ill-posed when the floating body is heaving in restricted water.

When we consider the computation of the sway added mass of a rectangular cylinder in a canal as shown in Fig. 1, we have the following potential-theory formulation in the zero-frequency limit

$$\nabla^2\phi(x,y)=0 \qquad \text{in } R \qquad (1a)$$

$$\phi_n=V\cdot n=-n_1 \qquad \text{on } S_0 \qquad (1b)$$

$$\phi_n=0 \qquad \text{on } S_F\cup S_w\cup S_B \qquad (1c)$$

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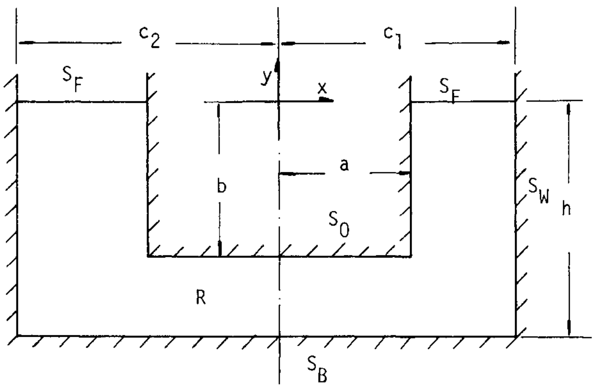


Fig. 1 Body configurations.

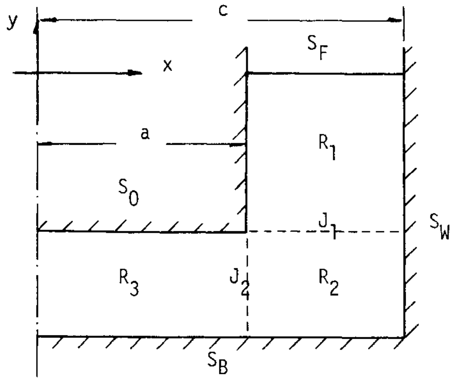


Fig. 2 The three subdomains of the symmetric configuration.

where $\partial R = S_0 \cup S_F \cup S_w \cup S_B$, $V=(1,0)$ is the unit sway velocity of the rectangular cylinder, and $\mathbf{n}=(n_1,n_2)$ is the unit outward normal vector. It is well-known that the solution of Eqs. (2a)-(2c) is not unique since there is an arbitrary additive constant. In order to insure uniqueness, we invoke the following Dirichlet condition at any one point $(x_0,y_0)$ in the fluid or on its boundary

$$\phi(x_0,y_0)=0 \qquad \text{at } (x_0 y_0)\epsilon(R\cup\partial R) \tag{1d}$$

Furthermore, we assume that the clearance between the body and the canal walls is small compared with the body dimensions. Here the rectangular body is not necessarily placed at the center of the rectangular canal. We assume

$$(c_1-a)/b=o(1) \tag{2a}$$

$$(c_2-a)/b=o(1) \tag{2b}$$

$$(h-b)/a=o(1) \tag{2c}$$

First, for simplicity, we consider the body at the center of the canal, i.e.

$$c=c_1=c_2 \tag{3}$$

Due to symmetry with respect to the $y$ axis, we shall consider only half of the original domain $x\geq 0$. Then we require along the symmetry line $(x=0)$

$$\phi(0,y)=0 \tag{4}$$

which replaces Eq. (1d).

We subdivide the fluid region $R(x\geq 0)$ into three subdomains, $R_1$, $R_2$, and $R_3$, as shown in Fig. 2, and require that the potential and the normal velocity be continuous across the juncture interfaces, $J_1$ and $J_2$, between adjacent subdomains. Let $\phi_1$ denote the velocity potential in $R_1$, $\phi_2$ in $R_2$ and $\phi_3$ in $R_3$. The conditions at the juncture boundaries $J_1$ and $J_2$ are

$$\phi_1=\phi_2 \tag{5a}$$

$$\phi_{1y}=\phi_{2y} \quad \text{on } J_1 \tag{5b}$$

$$\phi_2=\phi_3 \tag{6a}$$

$$\phi_{2x}=\phi_{3x} \quad \text{on } J_2 \tag{6b}$$

In the present Note we construct the particular solutions (with constants) in the form of $(x^2-y^2)+K$ or $x+K$ in each subdomain which satisfy the boundary conditions on $S_o$, $S_F$, $S_W$, $S_B$, and on the $y$ axis, and which also satisfy the continuity of the normal velocity across the juncture boundaries (5b) and (6b). They are

$$\phi_1=[(x-c)^2-y^2]/2(c-a)+K_1 \tag{7a}$$

$$\phi_2=-b[(x-c)^2-(y+h)^2]/[2(h-b)(c-a)]+K_2 \tag{7b}$$

$$\phi_3=bx/(h-b) \tag{7c}$$

where $K_1$ and $K_2$ are constants to be determined later. It is evident that the solution of Eqs. (1a)-(1c) and (4) can also be represented by a linear superposition of the complete set of homogeneous solutions in each element, in addition to the particular solutions given in Eq. (7) together with their coefficients to be determined later by imposing the juncture conditions (5) and (6). For example, the homogeneous solution in $R_1$, satisfies Laplace's equation in $R_1$, and a homogeneous Neuman condition on $S_o$, $S_F$, and $S_W$. However, we do not introduce the complete set of eigen-functions in each element even though it is very simple to do so. Instead we try to find the best approximations in terms of the forms given in Eq. (7). Since continuity of the normal velocity across interfaces $J_1$ and $J_2$ is already preserved, we find the constants $K_1$ and $K_2$ such that discontinuity of the potential along the juncture boundaries is minimum. One straightforward method of minimizing the potential jumps across $J_1$ and $J_2$ is the method of least squares as follows

$$\int_{J_1}(\phi_1-\phi_2)^2 dS=\min \tag{8a}$$

$$\int_{J_2}(\phi_2-\phi_3)^2 dS=\min \tag{8b}$$

By setting the derivatives of Eqs. (8a) and (8b) with respect to $K_1$ and $K_2$, respectively, to be zero, we have

$$\int_{J_1}(\phi_1-\phi_2)dS=0 \tag{9a}$$

$$\int_{J_2}(\phi_2-\phi_3)dS=0 \tag{9b}$$

The requirement specified in Eq. (9) is similar to the method of Lagrangian multipliers applied to the continuity condition of the potential across adjacent elements. By solving Eq. (9) in a straightforward manner, the constants $K_1$ and $K_2$ in Eq. (7) are obtained

$$K_1=\frac{b(2h+b)}{6(c-a)}+\frac{b(2a+c)}{3(h-b)}-\frac{c-a}{6}$$

$$K_2=\frac{b(a+c)}{2(h-b)}-\frac{b(h-b)}{6(c-a)} \tag{10}$$

**Table 1 The comparisons of the present results with the numerical results and with the results obtained from Newman's formula ($a/b = 1$)**

| | | Numerical solution | | Present formula | Newman's formula |
|---|---|---|---|---|---|
| $(c-a)/b$ | $(h-b)/a$ | Lower bound | Upper bound | Eq. (11) | Eq. (14) |
| 0.01 | 0.01 | 133.4705 | 133.9921 | 134.0033 | 136.3534 |
| 0.05 | 0.05 | 27.0400 | 27.3031 | 27.3500 | 29.7675 |
| 0.1 | 0.1 | 13.8072 | 13.9580 | 14.0333 | 16.5367 |
| 0.2 | 0.2 | 7.2167 | 7.2927 | 7.4000 | 10.0800 |
| 0.3 | 0.3 | 5.0311 | 5.0838 | 5.2111 | |
| 0.4 | 0.4 | 3.9493 | 3.9889 | 4.1333 | |
| 0.5 | 0.5 | 3.3058 | 3.3393 | 3.5000 | |
| 0.6 | 0.6 | 2.8819 | 2.9118 | 3.0889 | |
| 0.8 | 0.8 | 2.3623 | 2.3888 | 2.6000 | |
| 1.0 | 1.0 | 2.0602 | 2.0854 | 2.3333 | |
| 1.5 | 1.5 | 1.6784 | 1.7044 | 2.0556 | |
| 0.05 | 0.5 | 9.8295 | 10.0943 | 12.0500 | |
| 0.1 | 0.5 | 6.2497 | 6.3709 | 7.1000 | |
| 0.2 | 0.5 | 4.3806 | 4.4400 | 4.7000 | |
| 0.3 | 0.5 | 3.7584 | 3.8013 | 3.9667 | |
| 0.5 | 0.5 | 3.3058 | 3.3393 | 3.5000 | |
| 1.5 | 0.5 | 3.0550 | 3.0937 | 3.8333 | |
| 0.1 | 0.01 | 103.9777 | 105.1054 | 106.7333 | 105.4443 |
| 0.1 | 0.05 | 23.8183 | 24.0655 | 24.2000 | 25.9088 |
| 0.1 | 0.1 | 13.8072 | 13.9580 | 14.0333 | 16.5367 |
| 0.1 | 0.2 | 8.9165 | 9.0395 | 9.2000 | 12.9600 |
| 0.1 | 0.3 | 7.3690 | 7.4938 | 7.8111 | |
| 0.1 | 0.4 | 6.6528 | 6.7719 | 7.2833 | |
| 0.1 | 0.5 | 6.2497 | 6.3709 | 7.1000 | |
| 0.1 | 0.6 | 6.0010 | 6.1255 | 7.0889 | |
| 0.1 | 0.8 | 5.7257 | 5.8588 | 7.3250 | |
| 0.01 | 0.05 | 53.7210 | 54.6563 | 55.0700 | 60.6375 |
| 0.05 | 0.05 | 27.0400 | 27.3031 | 27.3500 | 29.7675 |
| 0.1 | 0.05 | 23.8183 | 24.0655 | 24.2000 | 25.9088 |
| 0.2 | 0.05 | 22.3794 | 22.6354 | 23.1500 | 23.9794 |
| 0.3 | 0.05 | 21.0025 | 22.2536 | 23.2667 | 23.3363 |
| 0.5 | 0.05 | 21.7661 | 22.0452 | 24.2000 | 22.8218 |
| 0.01 | 0.1 | 43.9984 | 45.1130 | 46.7033 | 56.4667 |
| 0.05 | 0.1 | 17.2524 | 17.3931 | 17.5167 | 20.9733 |
| 0.1 | 0.1 | 13.8072 | 13.9580 | 14.0333 | 16.5367 |
| 0.2 | 0.1 | 12.2536 | 12.3817 | 12.5667 | 14.3183 |
| 0.3 | 0.1 | 11.8200 | 11.9456 | 12.3222 | 13.5789 |
| 0.5 | 0.1 | 11.5608 | 11.6909 | 12.5667 | 12.9873 |

By substituting Eqs. (10) and (7), the added mass $\mu$ is given as a function of $a$, $b$, $c$, and $h$

$$\mu(a,b,c,h) = -\int_{S_0} n_i \phi \mathrm{d}S = \frac{m}{3a}\left\{\frac{bh}{c-a} + \frac{b(2a+c)}{h-b} + c - a\right\} \tag{11}$$

where $m$ is the mass of the fluid displaced by the body.

It is of interest to note that Eq. (11) may be extended to the case of an off-center rectangular cylinder located in a canal in a straightforward manner by neglecting the interaction between the fluid at each side of the body. Then we obtain, referring to Fig. 1

$$\mu = \tfrac{1}{2}[\mu(a,b,c_1,h) + \mu(a,b,c_2,h)] \tag{12}$$

Here, relation (2) is assumed to obtain Eq. (12). This formula can further be extended for the case when the left-hand side of the canal is located at infinity. Then the added mass of the rectangular cylinder with a single wall can be given under the assumptions given in Eqs. (2b) and (2c), as

$$\mu = \tfrac{1}{2}[\mu_\infty + \mu(a,b,c_1,h)] \tag{13}$$

where $\mu_\infty$ is the added mass in laterally unbounded water of finite depth $h$.

## Results and Discussion

The added mass of a rectangular cylinder ($a/b = 1$) at the center of a rectangular canal has been computed from the approximate formula (11). In Table 1, our results are shown with the numerical results obtained previously by the author and also with the results obtained by Newman,[3] who obtained an approximate formula for the added mass under the assumption given in Eq. (2) as

$$\mu = m\left(\frac{h^3}{3ab(c-a)} + \frac{h^2}{b(h-b)}\right) \tag{14}$$

The results obtained by Eq. (11) and those obtained by Eq. (14) are in good agreement with the numerical results obtained by Bai[1] for very small values of $(h-b)/a$ and $(c-a)/b$ in Table 1. However, the present formula gives a better approximation for a wider range of values of $(h-b)/a$ and $(c-a)/b$ than Newman's formula. The present formula gives a reasonable approximation of the added mass for

$$0 < (h-b)/a \leq 0.3 \qquad 0 < (c-a)/b \leq 0.3$$

It is of interest to note that our formula (11) reduces to Newman's formula (14) as $(h-b)/a$ and $(c-a)/b$ approach zero.

## Acknowledgment

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